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Topological Renormalisation of the O(3) Sigma Model

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Abstract

We show that the one-instanton sector moduli-space divergence of the O(3) Sigma Model leads to an unacceptable dependence of Green's functions on the arbitrary way that the field is split into a quantum fluctuation about a classical background. Since the divergence is associated with the degeneration of field configurations to those of the zero-instanton sector this arbitrariness may be cancelled by a 'topological counter-term' which we construct.

1 Introduction

Many field theories of physical interest have distinct topological sectors [1]. So, in the semi-classical approximation, Green's Functions are reduced to finite dimensional integrals over moduli that parametrise topologically non-trivial configurations [2]. Typically these integrals diverge, so to make sense of them we might introduce a cut-off in moduli space. How this is done depends on an arbitrary choice of how we split the field into classical background and quantum fluctuation. Amplitudes may then appear to be dependent on this choice. Clearly physical quantities should not depend on this arbitrary choice and so some procedure is required to ensure that this is the case. The purpose of this paper is to propose such a procedure.

For example, in the $O(3)$ σ-model the configuration space splits up into discrete pieces each characterised by a topological charge q [3]. Within each sector the action is minimised by instanton configurations [4]. The leading semi-classical contribution to the vacuum expectation value of some operator Λ can be written as [5]

$$\langle\Lambda\rangle = \sum_q \int \Lambda(a,b,c)\, \frac{K^q}{(q!)^2}\, e^{-h_q(a,b)}\, \frac{d^2c}{\pi(1+|c|^2)^2} \prod_j d^2a_j\; d^2b_j \tag{1}$$

where a, b, c are the moduli and are complex numbers, and $\Lambda(a,b,c)$ is the value of Λ evaluated at each instanton configuration. K is a coupling constant.

$$h_q(a,b) = -\sum_{i<j}^{q} \ln|a_i - a_j|^2 - \sum_{i<j}^{q} \ln|b_i - b_j|^2 + \sum_{i,j}^{q} \ln|a_i - b_j|^2 \tag{2}$$

so the partition function diverges as $a_i \to b_j$.

We shall show explicitly how simply cutting the integral off at $|a-b| = \epsilon$ makes this expectation value depend on the way that the field is split between the quantum piece and the classical background. This can be encoded into a "Ward Identity" analogous to that associated with BRST invariance in gauge theories [6].

The divergences are associated with classical configurations degenerating to configurations belonging to a topological sector with lower instanton number. This suggests the possibility of modifying the action in the sector with lower instanton number in such a way as to cancel the divergence, and hence the dependence on an arbitrary choice of quantization procedure.

The purpose of this paper is to show that this can be done for the $O(3)$ σ-model in the one instanton sector by adding a "counterterm" in the zero instanton sector. This "renormalisation" of the topological expansion is analogous to what occurs in Bosonic String Theory [7].

There are many similarities between the $O(3)$ σ-model, Yang-Mills Theory and Bosonic String Theory which make the σ-model a useful toy model. All three are classically conformally invariant [8], [9]. The σ-model and Yang-Mills Theory both display confinement, renormalisability, asymptotic freedom, and instantons characterised by integer-valued topological indices [10], [11] [12]. Other uses of the $O(3)$ σ-model are well known: i.e. it is equivalent to the CP^1 model [10] and it is relevant in describing the isotropic ferromagnet [1]. The $O(3)$ σ-model is particularly well-suited to our discussion since

the moduli-space divergence occurs at short distances and so is reliably computed in the semi-classical approximation. In Yang-Mills theory and String Theory, gauge-fixing is introduced by means of the Faddeev-Popov trick [13]. This may be extended so as to make the integration over the moduli explicit. An analogous method may be used in σ-models, such that the separation of the field into a classical background and quantum fluctuation may be done by imposing a constraint on the latter. If this is done with a delta function then the Faddeev-Popov method generates the appropriate moduli space measure. The constraint can be included in the action by means of ghost-like fields (which we shall call quasi-ghosts).

So the choice of constraint is important. There are popular choices, but these are by no means unique. In the development of Polyakov's formulation of Bosonic String Theory the metric degree of freedom to be integrated over is set equal to a fiducial metric depending on the moduli of a Riemann surface [14], [8]. In the semi-classical expansion of Yang-Mills theory [15] [16] two constraints are chosen, one being a background gauge condition, the other requiring that fluctuations of the gauge-field A about a classical solution $\mathcal{A}$ be orthogonal to the derivatives of the latter with respect to the moduli $\{t^A\}$ [7].

$$[\partial_\mu + \mathcal{A}_\mu, A_\mu - \mathcal{A}_\mu] = 0 \ \ , \ \ \int d^4x \ tr\left(\frac{\partial \mathcal{A}_\mu}{\partial t^A}\left(A - \mathcal{A}\right)_\mu\right) = 0 \tag{3}$$

Similarly, a natural, but not unique, choice of condition on fluctuations of the $O(3)$ σ-model field, leading to (1), is to demand that they are orthogonal to the derivatives of the classical instanton solution with respect to the moduli.

The construction of this paper is as follows. In Section 2 we look at the Green's Function for a field theory that admits instantons, and how we can separate the integral over the instanton moduli from that over the field. Our solution is to use a method analogous to that devised by Faddeev and Popov introducing a constraint on a quantum fluctuation about the instanton solution. We also see how the Green's Function is affected by varying it with respect to this constraint, finding that this results in an "anomalous Ward Identity." In section 3 we review the results of [5] which will be used in this paper. In section 4 we calculate the anomalous Ward Identity for the case of the one instanton solution in the $O(3)$ σ-model. It is found that in order to do the integral over the instanton moduli we must compactify our solution onto a sphere, a method for doing this is proposed.

Appendix A contains the detailed calculation for finding the two-point Green's Function of the fluctuation operator of the $O(3)$ σ-model. In Appendix B we describe the one instanton Kähler Metric [17] for the space of instanton moduli.

2 General Formulation

For a field theory in which the configuration space splits into distinct topological sectors, the partition function can be written as a sum of partition functions for each sector. Individual partition functions may be expressed as functional integrals over a particular

homotopy class. So for a theory with fields w and action $S[w]$

$$Z = \sum_q \kappa^q Z_q \ , \quad Z_q = \int_{\mathcal{C}_q} \mathcal{D}w \; e^{-S[w]} \tag{4}$$

where $\mathcal{C}_q$ are the distinct homotopy classes, labelled by integers q, of configurations of the fields w, and κ^q is some function of the topological coupling constant. The Green's Functions are then obtained as sums over the topological sectors

$$\mathcal{G}_q = \int_{\mathcal{C}_q} \mathcal{D}w \; e^{-S[w]} \; \Lambda(w) \tag{5}$$

$$\mathcal{G} = \frac{1}{Z} \sum_q \kappa^q \mathcal{G}_q \tag{6}$$

In each class there is in general a family of solutions, v, to the classical equations of motion which are parametrized by moduli $\{t^A\}$. Thus

$$\left.\frac{\delta S}{\delta w}\right|_{w=v} = 0 \qquad \text{for all } \mathrm{t^A} \tag{7}$$

Differentiating with respect to t^A shows that $\frac{\partial v}{\partial t^A}$ is a zero mode of the fluctuation operator

$$\left.\frac{\delta^2 S}{\delta w \delta w}\right|_{w=v} \tag{8}$$

If we set $w = v + \phi$ where ϕ is a quantum fluctuation continuously deformable to zero then the action may be expanded in powers of ϕ. Fluctuations of ϕ in the directions of the zero modes do not contribute to the action to leading order and so are not exponentially damped. They correspond to variations of the moduli. To separate the integrals over t from the integrals over ϕ, we might try to choose ϕ to be orthogonal to the zero modes if this is possible, but this is arbitrary. More generally, we could introduce some arbitrary constraint on ϕ to specify that the fluctuations are transverse to the space of zero modes. We do this by means of a set of functions $F_j(w,t) = 0$, which are introduced into the Green's Function in a manner analogous to the Faddeev-Popov method, i.e. by multiplying the Green's Function by

$$\int dt \; \Delta[w,t] \prod_j \delta\left(F_j(w,t)\right) = 1 \tag{9}$$

Then we can represent the constraint in the action by using ghosts. Suppose that the constraints have a solution $w = \hat{w}$ and $t = \hat{t}$, then expanding the constraint about $t = \hat{t} + \tilde{t}$

$$F_j(w,t) = F_j(\hat{w},\hat{t}) + \tilde{t}^A \partial_A F_j(w,t)|_{w=\hat{w},t=\hat{t}} + \cdots \tag{10}$$

where $\partial_A = \frac{\partial}{\partial t^A}$. So

$$\int d\tilde{t} \; \Delta[w,t] \prod_j \delta\left(\tilde{t}^A \partial_A F_j(w,t)|_{w=\hat{w},t=\hat{t}}\right) = 1 \tag{11}$$

and we may factor out the operator Δ. Integrating out the delta function leaves the Jacobian, thus

$$\Delta = \det\left(\partial_A F_j(w,t)|_{w=\hat{w},t=\hat{t}}\right) \tag{12}$$

Representing this determinant using Grassmann numbers we have a quasi-anti-ghost ξ^j for each constraint F_j, and a quasi-ghost τ^A for each parameter t^A. Then

$$\Delta = \int d\xi\, d\tau\ \exp\left[-\tau^A \partial_A\left(\xi^j F_j(w,t)\right)\right] \tag{13}$$

Also, if we write

$$\prod_j \delta\left(F_j(w,t)\right) = \int d\lambda\ e^{-i\lambda^j F_j} \tag{14}$$

then we get

$$\mathcal{G}_q = \int dt\ \mathcal{D}W\ e^{-S_{tot}}\ \Lambda(w) \tag{15}$$

where

$$S_{tot} = S[w] + \tau^A \partial_A\left(\xi^B F_B(w,t)\right) + i\lambda^j F_j \tag{16}$$

and W denotes w, ξ, λ. We may write this more usefully by using a 'BRST' transformation. If the transformation is parametrised by a Grassmann number η, and acts on w, ξ, λ but not the moduli, then it may be written as $\delta_\eta w = \eta\varsigma w$, where the operator ς is defined by

$$\varsigma\xi^B = i\lambda^B\ ,\ \varsigma\lambda = 0\ ,\ \varsigma\tau = 0\ ,\ \varsigma w = 0\ ,\ \varsigma^2 = 0 \tag{17}$$

So we have

$$S_{tot} = S[w] + (\varsigma + \tau^A\partial_A)(\xi^j F_j(w,t)) \tag{18}$$

Now $\{\varsigma, \tau^A\partial_A\} = 0$ because ∂_A acts only on the moduli whereas ς does not. Also $\left(\tau^A\partial_A\right)^2 = 0$ as derivatives with respect to the moduli commute with each other. Thus $\varsigma + \tau^A\partial_A$ is also nilpotent. Given that $S[w]$ is independent of the moduli we thus have

$$(\varsigma + \tau^A\partial_A)S_{tot} = 0 \tag{19}$$

So the action is not 'BRST' invariant but $\varsigma S_{tot} = -\tau^A\partial_A S_{tot}$.

We have now achieved our purpose of being able to write the Green's Function as an integral over the moduli explicitly

$$\mathcal{G}_q = \int dt\ g(t)\ \ ,\ \ g(t) = \int \mathcal{D}W\ e^{-S_{tot}}\ \Lambda(w) \tag{20}$$

If these integrals diverge then they may be regulated by a cut-off procedure. However we need to find out if the choice of constraint has any effect on this. If it does then this is obviously unsatisfactory and we must do something to compensate. Making an arbitrary variation of the moduli density with respect to the constraint gives

$$\begin{aligned}
\delta g(t) &= -\int \mathcal{D}W\ e^{-S_{tot}}(\varsigma + \tau^A\partial_A)(\xi^j\delta F_j)\ \Lambda(w) \\
&= -\int \mathcal{D}W\ (\varsigma + \tau^A\partial_A)\ e^{-S_{tot}}\ (\xi^j\delta F_j)\ \Lambda(w)
\end{aligned} \tag{21}$$

because of (19). But $\int \mathcal{D}W \varsigma\ e^{-S_{tot}}\ (\xi^j \delta F_j)\ \Lambda(w)$ vanishes as $\int \mathcal{D}W\ e^{-S_{tot}}\ (\xi^j \delta F_j)\ \Lambda(w)$ is Grassmann odd. Then

$$\delta g(t) = -\partial_A \int \mathcal{D}W\ e^{-S_{tot}} \tau^A (\xi^j \delta F_j)\ \Lambda(w) \tag{22}$$

If the integration region of the parameters t is M which has boundary ∂M then we may calculate the variation in the Partition Function by using Stokes' Theorem in the form

$$\int_M d^n t\ \frac{\partial}{\partial t^A} \left(f^A(t) \right) = \int_{\partial M} d\Sigma_A \left(f^A(t) \right) \tag{23}$$

So

$$\delta \mathcal{G}_q = \delta \int_M dt\ g(t) = -\int_{\partial M} d\Sigma_A \int \mathcal{D}W\ \tau^A (\xi^j \delta F_j) e^{-S_{tot}}\ \Lambda(w) \tag{24}$$

If this is non-zero then the Green's Function has aquired a dependence on the arbitrary choice F. This is unacceptable and we must find some way to cure this problem. Typically the divergencies are associated with a classical configuration degenerating to one of lower topology. So configurations on the boundary ∂M may be approximated by configurations in a different topological sector. Thus (24) may possibly be cancelled by a counterterm from another topological sector. We shall see that this kind of 'topological renormalisation' can be done in the case of the $O(3)$ σ-model, at least for the simplest kind of divergences.

3 One loop expansion of the $O(3)$ σ-model

In this section we shall review the results of [5] (further details may be found in [18]). The action of the two dimensional static $O(3)$ σ-model is

$$S = \frac{1}{2k} \int \sum_{a=1}^{3} (\partial_\mu \sigma^a)^2 d^2 x \tag{25}$$

where there are three scalar fields $\sigma^a(x, y)$ obey $\sum_{a=1}^3 \sigma^a \sigma^a = \sigma.\sigma = 1$ and $\mu = x, y$. k is the coupling constant. This constraint can be included in the action by means of a Lagrange Multiplier which has on shell value $l = \sigma.\partial^2\sigma$. Thus the model has field equations

$$\partial^2 \sigma - \left(\sigma.\partial^2 \sigma \right) \sigma = 0 \tag{26}$$

The finite action solutions, $\sigma^{(0)}(x, y)$, tend to a constant as x and y tend to infinity. Thus it is possible to compactify the coordinate space plane R^2 on to a sphere which we shall call S^2_{phys}. Also the space of fields σ^a, subject to $\sigma.\sigma = 1$, is a spherical surface of unit radius. We shall call this 'internal space' S^2_{int}.

The finite-action configurations of σ (instantons) are then mappings of S^2_{phys} to S^2_{int} and thus can be classified into homotopy classes. Instanton solutions arise at the minima of the action (not zero), i.e. when $S = \frac{4\pi q}{k}$ where q is the integer valued topological charge which characterises the homotopy sectors.

It is convenient to stereographically project the spheres S^2_{int} and S^2_{phys} onto planes. The field on the plane is

$$w(z,\bar{z}) = \frac{\sigma^1 + i\sigma^2}{1+\sigma^3} \tag{27}$$

where $z = x + iy$ is the complex coordinate on the stereographic projection of S^2_{phys}. In this notation the action (25) becomes

$$S = \frac{4}{k}\int d^2x\, \frac{\partial_z w \partial_{\bar{z}} \bar{w} + \partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{28}$$

where $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. The topological charge on the plane is given by

$$q = \frac{1}{\pi}\int d^2x\, \frac{\partial_z w \partial_{\bar{z}} \bar{w} - \partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{29}$$

which means that

$$S = \frac{4\pi q}{k} + \frac{8}{k}\int d^2x\, \frac{\partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} \tag{30}$$

and the instantons are given by solutions to $\partial_{\bar{z}} w = 0$. Thus we may write the q-instanton solution as

$$w = v = c\left(\frac{(z-a_1)\cdots(z-a_q)}{(z-b_1)\cdots(z-b_q)}\right) \tag{31}$$

Note that the number of poles in the instanton is also equal to the topological charge. Taking the limit $a \to b$ in a particular sector removes one of these poles and so we move to a solution with a topological charge lower by one.

For the one instanton case

$$w = v = c\left(\frac{z-a}{z-b}\right) \tag{32}$$

the vacuum expectation value given in (1), (2) simplifies

$$\begin{aligned} \langle\Lambda\rangle_1 &= \int \Lambda(a,b,c)\, K^1 \,\frac{1}{|a-b|^2}\, \frac{d^2c}{\pi(1+|c|^2)^2}\, d^2a\, d^2b \\ &= \int \Lambda(a,b,c)\, \zeta_1(a,b,c)\, d^2c\, d^2a\, d^2b \end{aligned} \tag{33}$$

where $K^1 = 2^6 e^{\gamma-2}$, $\gamma = 0.5772$ is the Euler number. This is calculated in [5] by the steepest descent method. A quantum fluctuation about the instanton solution v is defined and is chosen to be orthogonal to the zero modes of the fluctuation operator. The integrand $\zeta_1(a,b,c)$ is divergent as $a \to b$, i.e. as we approach the zero instanton sector from the one instanton sector.

In the zero instanton case $v = c$ and $K^0 = 1$, so

$$\langle\Lambda\rangle_0 = \int d^2c\, \frac{\Lambda(c)}{\pi(1+|c|^2)^2} \tag{34}$$

4 Anomalous Ward Identity

We shall now calculate the 'anomaly' given in (24) for the case of the $O(3)$ σ-model. If we set $a-b=r$ the one instanton solution becomes

$$v = c\left(1 - \frac{r}{z-b}\right) \tag{35}$$

then as $|r|$ approaches zero, v approaches the zero instanton configuration $v=c$, so we will introduce a moduli-space cut-off by taking $|r|>\epsilon$. We take ϵ, b and c as the complex instanton parameters, defining them collectively as $\{t^{\alpha}\} = (r,b,c)$ and $\{t^{\bar{\alpha}}\} = (\bar{r},\bar{b},\bar{c})$. Note that w is independent of $\{t^{\alpha}\}$. Now suppose that w differs from v by some quantum correction $\varphi(z,\bar{z})$, then $w=v+\varphi$. The quantum correction is constrained to be orthogonal to a set of arbitrarily chosen functions $\{\psi\}$

$$F_{\bar{\alpha}} \equiv (\psi_{\alpha},\varphi) = 0 = (\bar{\psi}_{\bar{\alpha}},\bar{\varphi}) \equiv F_{\alpha} \tag{36}$$

The inner product is defined as

$$(\psi_{\alpha},\varphi) = \int d^2x\,\sqrt{g}\,\rho^{-2}\,\bar{\psi}_{\bar{\alpha}}\,\varphi \quad , \quad \rho = 1+|v|^2 \tag{37}$$

where $g_{\mu\nu}$ is the metric on S^2_{phys}, and $g=\det g_{\mu\nu}$. The action (30) may be expanded in orders of φ. To leading order it is

$$S_0[\varphi] = \frac{4\pi q}{k} + \frac{8}{k}(\varphi,\Delta\varphi) \tag{38}$$

where the fluctuation operator is

$$\Delta = -\frac{\rho^2}{\sqrt{g}}\partial_z\rho^{-2}\partial_{\bar{z}} \tag{39}$$

where $\partial_z = \frac{\partial}{\partial\bar{z}}$. This is different to the fluctuation operator used in [5], but this is just because the variables in the second term of the action are defined differently in each case. It is easy to show that the fluctuation operators are equivalent.

The functionals F_A are the constraints which enter into the action in the manner outlined in the Section 2. Therefore the total action (18) is

$$S_{tot} = S_0[\varphi] + (\varsigma + \tau^A\partial_A)(\xi^B F_B) \tag{40}$$

and the 'anomaly' (24)

$$\delta\mathcal{G}_1 = -\int_{\partial M} d\Sigma_A \int \mathcal{D}\Phi\; e^{-S_{tot}}\tau^A(\xi^B\delta F_B)\,\Lambda(v) \tag{41}$$

where Φ denotes all the fields. Looking to integrate out the ghosts we expand out the action using the properties of the operator ς given in (17)

$$\begin{aligned} S_{tot} &= S_0[\varphi] + \lambda^A F_A \quad -\tau^{\alpha}\xi^{\bar{\beta}}m_{\alpha\bar{\beta}} - \tau^{\bar{\alpha}}\xi^{\beta}\bar{m}_{\bar{\alpha}\beta} \\ &\qquad +\tau^{\alpha}\xi^{\beta}(\varphi,\partial_{\alpha}\psi_{\beta}) + \tau^{\alpha}\xi^{\bar{\beta}}(\bar{\varphi},\partial_{\alpha}\bar{\psi}_{\bar{\beta}}) \\ &\qquad +\tau^{\bar{\alpha}}\xi^{\beta}(\varphi,\partial_{\bar{\alpha}}\psi_{\beta}) + \tau^{\bar{\alpha}}\xi^{\bar{\beta}}(\bar{\varphi},\partial_{\bar{\alpha}}\bar{\psi}_{\bar{\beta}}) \end{aligned} \tag{42}$$

The ghost field propagators have been written as

$$m_{\alpha\bar{\beta}} = \left(\partial_{\bar{\alpha}}\bar{v}, \bar{\psi}_{\bar{\beta}}\right) \quad , \quad \bar{m}_{\bar{\alpha}\beta} = \left(\partial_{\alpha}v, \psi_{\beta}\right) \tag{43}$$

The last four terms of (42) are the interaction terms between the ghost fields and φ. Now let $\tilde{S} = S_0 + \lambda^A F_A - \tau^{\alpha}\xi^{\bar{\beta}} m_{\alpha\bar{\beta}} - \tau^{\bar{\alpha}}\xi^{\beta}\bar{m}_{\bar{\alpha}\beta}$ and expand the rest of the action in a power series in φ. So if $S_{tot} = \tilde{S} + \bar{S}$ then

$$e^{-\bar{S}} = 1 - \left[\tau^M \xi^{\nu}(\varphi, \partial_M \psi_{\nu}) + \tau^M \xi^{\bar{\nu}}(\bar{\varphi}, \partial_M \bar{\psi}_{\bar{\nu}})\right] + \cdots \tag{44}$$

M stands for both μ and $\bar{\mu}$. The first term in the expansion disappears under contraction with the rest of $\mathcal{G}_1$ as it is linear in φ. The terms to leading order in the coupling are

$$\begin{aligned}
\delta\mathcal{G}_1 &= -\int_{\partial M} d\Sigma_{\alpha} \int \mathcal{D}\Phi\; e^{-\tilde{S}} \left(\tau^{\alpha}\xi^{\beta}(\varphi, \delta\psi_{\beta})\tau^{\bar{\mu}}\xi^{\bar{\nu}}(\bar{\varphi}, \partial_{\bar{\mu}}\bar{\psi}_{\bar{\nu}})\; \Lambda(v)\right) \\
&\qquad - \int_{\partial M} d\Sigma_{\bar{\alpha}} \int \mathcal{D}\Phi\; e^{-\tilde{S}} \left(\tau^{\bar{\alpha}}\xi^{\beta}(\varphi, \delta\psi_{\beta})\tau^{\mu}\xi^{\bar{\nu}}(\bar{\varphi}, \partial_{\mu}\bar{\psi}_{\bar{\nu}})\; \Lambda(v)\right) \\
&= -\int_{\partial M} d\Sigma_{\alpha} \left(\zeta_1(t)\bar{m}^{-1}_{\bar{\mu}\beta}(\boldsymbol{\varphi}, \delta\psi_{\beta}) m^{-1}_{\alpha\bar{\nu}}(\bar{\boldsymbol{\varphi}}, \partial_{\bar{\mu}}\bar{\psi}_{\bar{\nu}})\; \Lambda(v)\right) \\
&\qquad - \int_{\partial M} d\Sigma_{\bar{\alpha}} \left(\zeta_1(t)\bar{m}^{-1}_{\bar{\alpha}\beta}(\boldsymbol{\varphi}, \delta\psi_{\beta}) m^{-1}_{\mu\bar{\nu}}(\bar{\boldsymbol{\varphi}}, \partial_{\mu}\bar{\psi}_{\bar{\nu}})\; \Lambda(v)\right) \\
&= -\int_{\partial M} d\Sigma_{\alpha}\Psi^{\alpha}\; \Lambda(v) - \int_{\partial M} d\Sigma_{\bar{\alpha}}\Psi^{\bar{\alpha}}\; \Lambda(v)
\end{aligned} \tag{45}$$

Where $\zeta_1(t)$ is the one-loop partition function given above in (33), and the boldface type indicates that the field $\boldsymbol{\varphi}$ is contracted with its conjugate field in the same term. We will now show that $\Psi^{\bar{\alpha}}$ and Ψ^{α} can themselves be expressed as variations. This enables us to both simplify their evaluations and find counterterms which will cancel $\delta\int_M dt\; g(t)$. To do this we see that the fields $\boldsymbol{\varphi}$ contract to introduce a two-point Greens function. The Greens function of the two-point fluctuation operator is found in Appendix A to be

$$\mathcal{I}(x,y) = \;<\varphi(x)\;\bar{\varphi}(y)> = \boldsymbol{\varphi}(x)\bar{\boldsymbol{\varphi}}(y) \tag{46}$$

$$= -\frac{1}{\pi^2}\int d^2z\; (1 - P^{\dagger}.)\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}}(1 - .P) \tag{47}$$

here we have used a dot notation equivalent to the inner product, i.e.

$$P^{\dagger}(y,x).\frac{1}{x-z} = \int d^2x'\; P^{\dagger}(y,x')\;\frac{1}{x'-z}\;\sqrt{g}\;\rho^{-2}(x') \tag{48}$$

where

$$P(x,y) = \psi_a(x)\bar{m}^{-1}_{\bar{b}a}\bar{\mathcal{Z}}_{\bar{b}}(y) \tag{49}$$

$$P^{\dagger}(y,x) = \mathcal{Z}_b(y)m^{-1}_{b\bar{a}}\bar{\psi}_{\bar{a}}(x) \tag{50}$$

and $\mathcal{Z}(x)$ is a zero mode of Δ. Above we showed that the zero modes of the fluctuation operator are $\partial_A v$. If we denote these by $\mathcal{Z}(x)$ then from (43) we see that

$$m_{\alpha\bar{\beta}} = \left(\bar{\mathcal{Z}}_{\bar{\alpha}}, \bar{\psi}_{\bar{\beta}}\right) \quad , \quad \bar{m}_{\bar{\alpha}\beta} = \left(\mathcal{Z}_{\alpha}, \psi_{\beta}\right) \tag{51}$$

We are interested in Ψ evaluated on the cut-off boundary ∂M given by $|a-b| = |r| = \epsilon$. We show in Appendix A that here the Green's Function $\mathcal{I}(x,y)$ can be expressed in terms of the zero instanton sector Green's Function as

$$\mathcal{I}(x,y) = (1 - P^\dagger.)\, \mathcal{I}_0(x,y)\, (1 - .P) \times (1 + O(\epsilon)) \tag{52}$$

If we denote $\mathcal{I}_0(x,y)$ by the zero instanton sector contraction $\bar{\boldsymbol{\varphi}}_0(x) \quad \boldsymbol{\varphi}_0(y)$ we can approximate Ψ in terms of the zero instanton sector as follows. Applying the appropriate parts of Ψ^α to $\mathcal{I}(x,y)$ and again using the dot notation

$$(\varphi, \delta\psi_\beta) = \int d^2x\, \sqrt{g}\, \rho^{-2}\, \bar{\varphi}\, \delta\psi_\beta = \bar{\varphi} \cdot \delta\psi_\beta \tag{53}$$

gives for f an arbitrary function independent of ψ

$$\begin{aligned}
\bar{m}^{-1}_{\bar{\mu}\beta} \Big((f(1 - \cdot P)) \cdot \delta\psi_\beta \Big) &= \bar{m}^{-1}_{\bar{\mu}\beta} \Big(f \cdot \delta\psi_\beta \Big) - \bar{m}^{-1}_{\bar{\mu}\beta} \, (f \cdot \psi_a)\, \bar{m}^{-1}_{\bar{b}a} \Big(\bar{\mathcal{Z}}_{\bar{b}} \cdot \delta\psi_\beta \Big) \\
&= \bar{m}^{-1}_{\bar{\mu}\beta} \Big(f \cdot \delta\psi_\beta \Big) - \bar{m}^{-1}_{\bar{\mu}\beta} \, (f \cdot \psi_a)\, \bar{m}^{-1}_{\bar{b}a} \delta\bar{m}_{\bar{b}\beta} \\
&= \bar{m}^{-1}_{\bar{\mu}\beta} \Big(f \cdot \delta\psi_\beta \Big) - (f \cdot \psi_a)\, \delta\bar{m}^{-1}_{\bar{\mu}a} \\
&= \delta(f \cdot \psi_\beta \bar{m}^{-1}_{\bar{\mu}\beta})
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
m^{-1}_{\alpha\bar{\nu}} \left(\Big((1 - P^\dagger \cdot) f \Big) \cdot \frac{\partial}{\partial t^{\bar{\mu}}} \bar{\psi}_{\bar{\nu}} \right) &= m^{-1}_{\alpha\bar{\nu}} \left[\frac{\partial}{\partial t^{\bar{\mu}}} \Big(\Big((1 - P^\dagger \cdot) f \Big) \cdot \bar{\psi}_{\bar{\nu}} \Big) \right. \\
&\qquad \left. - \left(\left(\frac{\partial}{\partial t^{\bar{\mu}}} \Big((1 - P^\dagger \cdot) f \Big) \right) \cdot \bar{\psi}_{\bar{\nu}} \right) \right] \\
&= m^{-1}_{\alpha\bar{\nu}} \left(\frac{\partial P^\dagger}{\partial t^{\bar{\mu}}} \cdot f \right) \cdot \bar{\psi}_{\bar{\nu}} \\
&= m^{-1}_{\alpha\bar{\nu}} \left(\frac{\partial}{\partial t^{\bar{\mu}}} \Big(\mathcal{Z}_b m^{-1}_{b\bar{a}} \bar{\psi}_{\bar{a}} \Big) \cdot f \right) \cdot \bar{\psi}_{\bar{\nu}} \\
&= m^{-1}_{\alpha\bar{\nu}} m_{b\bar{\nu}} \left(\frac{\partial}{\partial t^{\bar{\mu}}} \Big(m^{-1}_{b\bar{a}} \bar{\psi}_{\bar{a}} \Big) \cdot f \right) \\
&= \frac{\partial}{\partial t^{\bar{\mu}}} m^{-1}_{\alpha\bar{a}} \bar{\psi}_{\bar{a}} \cdot f
\end{aligned} \tag{55}$$

Similarly

$$\bar{m}^{-1}_{\bar{\alpha}\beta} \Big((f(1 - .P))\, .\delta\psi_\beta \Big) = \delta(f.\psi_\beta \bar{m}^{-1}_{\bar{\alpha}\beta}) \tag{56}$$

$$m^{-1}_{\mu\bar{\nu}} \left(\Big((1 - P^\dagger.) f \Big) . \frac{\partial}{\partial t^\mu} \bar{\psi}_{\bar{\nu}} \right) = \frac{\partial}{\partial t^\mu} m^{-1}_{\bar{a}\mu} \bar{\psi}_{\bar{a}} .f \tag{57}$$

so that

$$\begin{aligned}
\Psi^\alpha &= \zeta_1(t) \delta \Big[\bar{m}^{-1}_{\bar{\mu}\beta} (\boldsymbol{\varphi}_0, \psi_\beta) \Big] \frac{\partial}{\partial t^{\bar{\mu}}} (m^{-1}_{\bar{a}\alpha} \bar{\boldsymbol{\varphi}}_0, \bar{\psi}_{\bar{a}}) \\
&= \delta \left[\zeta_1(t) \bar{m}^{-1}_{\bar{\mu}\beta} (\boldsymbol{\varphi}_0, \psi_\beta) \frac{\partial}{\partial t^{\bar{\mu}}} (m^{-1}_{\bar{a}\alpha} \bar{\boldsymbol{\varphi}}_0, \bar{\psi}_{\bar{a}}) \right]
\end{aligned} \tag{58}$$

and

$$
\begin{aligned}
\Psi^{\bar{\alpha}} &= \zeta_1(t)\delta\left[\bar{m}^{-1}_{\bar{\alpha}\beta}(\boldsymbol{\varphi}_0,\psi_\beta)\right]\frac{\partial}{\partial t^\mu}(m^{-1}_{\bar{a}\mu}\bar{\boldsymbol{\varphi}}_0,\bar{\psi}_{\bar{a}}) \\
&= \delta\left[\zeta_1(t)\bar{m}^{-1}_{\bar{\alpha}\beta}(\boldsymbol{\varphi}_0,\psi_\beta)\frac{\partial}{\partial t^\mu}(m^{-1}_{\bar{a}\mu}\bar{\boldsymbol{\varphi}}_0,\bar{\psi}_{\bar{a}})\right] \qquad (59)
\end{aligned}
$$

Thus we have succeeded in writing Ψ^α and $\Psi^{\bar{\alpha}}$ on ∂M as variations, given in terms of the zero instanton sector Green's Function. We will now write this as a zero instanton sector expectation value. In this sector

$$
\int_0 \mathcal{D}\Phi\, e^{-S_{tot}}\ \bar{\varphi}_0(x)\varphi_0(y) = \int K^0\ \bar{\boldsymbol{\varphi}}_0(x)\boldsymbol{\varphi}_0(y)\ \frac{d^2c}{\pi(1+|c|^2)^2} \qquad (60)
$$

to leading order in the coupling. The one instanton moduli are b, c and $r=a-b$ whose magnitude is held fixed on ∂M at $|r|=\epsilon$, so we can set

$$
(d\Sigma^A) = (d\Sigma^r, d\Sigma^{\bar{r}}) = (d\bar{r}\ d^2b\ d^2c\ ,\ -dr\ d^2b\ d^2c) \qquad (61)
$$

and as $\zeta_1(t) = \frac{K^1}{\pi|r|^2(1+|c|^2)^2}$ hence

$$
\begin{aligned}
\delta\mathcal{G}_1 &= \delta\int_0 \mathcal{D}\Phi\, e^{-S_{tot}}\left(\oint_{|r|=\epsilon} d\bar{r}\ d^2b\ \frac{K^1}{|r|^2}\left(\bar{m}^{-1}_{\bar{\mu}\beta}(\varphi_0,\psi_\beta)\frac{\partial}{\partial t^{\bar{\mu}}}(m^{-1}_{\bar{a}r}\bar{\varphi}_0,\bar{\psi}_{\bar{a}})\right)\right. \\
&\quad \left. -\oint_{|r|=\epsilon} dr\ d^2b\ \frac{K^1}{|r|^2}\left(\bar{m}^{-1}_{\bar{r}\beta}(\varphi_0,\psi_\beta)\frac{\partial}{\partial t^\mu}(m^{-1}_{\bar{a}\mu}\bar{\varphi}_0,\bar{\psi}_{\bar{a}})\right)\right)\Lambda(c) \qquad (62) \\
&\equiv \delta\int_0 \mathcal{D}\Phi\, e^{-S_{tot}}\ \mathcal{J}\ \Lambda(c) \qquad (63)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{J} &= \int d^2b\ \left[\oint_{|r|=\epsilon} d\bar{r}\ \frac{K^1}{|r|^2}\left(\bar{m}^{-1}_{\bar{\mu}\beta}(\varphi_0,\psi_\beta)\frac{\partial}{\partial t^{\bar{\mu}}}(m^{-1}_{\bar{a}r}\bar{\varphi}_0,\bar{\psi}_{\bar{a}})\right)\right. \\
&\quad \left. -\oint_{|r|=\epsilon} dr\ \frac{K^1}{|r|^2}\left(\bar{m}^{-1}_{\bar{r}\beta}(\varphi_0,\psi_\beta)\frac{\partial}{\partial t^\mu}(m^{-1}_{\bar{a}\mu}\bar{\varphi}_0,\bar{\psi}_{\bar{a}})\right)\right] \qquad (64)
\end{aligned}
$$

in terms of the zero instanton sector fields. Since we have succeeded in writing the variation of the one instanton sector contribution as the variation of a zero instanton contribution, it is clear that we can cancel the divergence from this variation by a counterterm in the zero instanton sector. We will thus modify (6) by defining

$$
\tilde{\mathcal{G}} = \frac{1}{\tilde{Z}}\sum_q \kappa^q\tilde{\mathcal{G}}_q \qquad (65)
$$

where

$$
\begin{aligned}
\tilde{\mathcal{G}}_0 &= \int \mathcal{D}w\ e^{-S[w]}\ \Lambda(w)\ (1-\kappa\mathcal{J}) \qquad (66) \\
\tilde{Z}_0 &= \int \mathcal{D}w\ e^{-S[w]}\ (1-\kappa\mathcal{J}) \qquad (67) \\
\tilde{\mathcal{G}}_1 &= \mathcal{G}_1\ ,\ \ \tilde{Z}_1 = Z_1 \qquad (68)
\end{aligned}
$$

so that to the order that we are working

$$\delta_\psi \tilde{\mathcal{G}} = 0 \tag{69}$$

Clearly to higher orders there are further modifications. We will now evaluate $\mathcal{J}$ for the natural choice of the constraint F which is when the ψ are the zero modes $\frac{\partial v}{\partial t}$ of the fluctuation operator. In this case the propagators of the ghost fields (43) become the Kähler metric which is the metric tensor on the manifold of the instantons, parametrized by the $\{t^\alpha\}$. $m_{\alpha\bar{\beta}}$ is Kähler as it can be written in the form [17],[18]

$$m_{\alpha\bar{\beta}} = \frac{\partial}{\partial t^\alpha}\frac{\partial}{\partial t^{\bar{\beta}}}\mathcal{K} \ \ , \ \ \mathcal{K} = \int d^2x\, \sqrt{g(x)}\, \ln(1+|v|^2) \tag{70}$$

where $\mathcal{K}$ is the Kähler Potential. $m_{\alpha\bar{\beta}}$ can be calculated for the one instanton case (see Appendix B). Henceforth $m_{\alpha\bar{\beta}}$ shall mean the Kähler metric. To calculate $\mathcal{J}$ we have for $v = c\left(1 - \frac{r}{z-b}\right)$

$$\psi_\beta = \frac{\partial v}{\partial t^\beta} = \begin{pmatrix} \frac{\partial v}{\partial r} \\ \\ \frac{\partial v}{\partial b} \\ \\ \frac{\partial v}{\partial c} \end{pmatrix}_\beta = \begin{pmatrix} -\frac{c}{z-b} \\ \\ -\frac{cr}{(z-b)^2} \\ \\ 1 - \frac{r}{z-b} \end{pmatrix}_\beta \tag{71}$$

and $m^{-1}_{\bar{\alpha}\beta}$ is the inverse of the Kahler metric

$$\left(m^{-1}_{\bar{\alpha}\beta}\right) = \frac{1}{m}\begin{pmatrix} A & B & C \\ B^\dagger & D & E \\ C^\dagger & E^\dagger & F \end{pmatrix} \equiv \left(\frac{m'_{\bar{\alpha}\beta}}{m}\right) \tag{72}$$

where A, B, C, D, E and F, are all functions of the $\{t^\alpha\}$ (see Appendix B), and $m = \det(m_{\bar{\alpha}\beta})$. Also

$$\left(\bar{m}^{-1}_{\bar{\alpha}\beta}\right) = \frac{\left(m'_{\bar{\alpha}\beta}\right)^T}{\bar{m}} \tag{73}$$

both terms in $\mathcal{J}$ may be split into a couple of inner products which may be treated separately. We need to calculate

$$\bar{m}^{-1}_{\bar{\mu}\beta}\,(\varphi_0, \psi_\beta) = \int d^2z \sqrt{g}\, \rho^{-2}\, \bar{m}^{-1}_{\bar{\mu}\beta}\, \psi_\beta\, \bar{\varphi}_0(z,\bar{z}) \tag{74}$$

where

$$\rho^{-2} = \frac{1}{(1+|v|^2)^2} = \frac{|z-b|^4}{(|z-b|^2 + |c|^2|z-b-r|^2)^2} \tag{75}$$

and the square root of the determinant of the metric on S^2_{phys} is

$$\sqrt{g} = \frac{1}{(1+|z|^2)^2} \tag{76}$$

So

$$\bar{m}^{-1}_{\bar{\mu}\beta}\left(\varphi_0,\psi_\beta\right) = \frac{\left(m'_{\bar{\alpha}\beta}\right)^T}{\bar{m}}\int d^2z\,\frac{|z-b|^4}{(1+|z|^2)^2(|z-b|^2+|c|^2|z-b-r|^2)^2} \times \begin{pmatrix} -\frac{c}{z-b} \\ \\ -\frac{cr}{(z-b)^2} \\ \\ 1-\frac{r}{z-b}\end{pmatrix}_\beta \bar{\varphi}_0(z,\bar{z}) \tag{77}$$

However this needs to be evaluated on the plane so that we are able to use the results of [5] which are all given on the plane. To do this we simply replace g with the flat space metric. These three integrals don't need to be evaluated until later. However we notice at this stage that one of the integrals is zero for small r

$$\int d^2z\,\frac{|z-b|^4\bar{\varphi}_0(z,\bar{z})}{(|z-b|^2+|c|^2|z-b-r|^2)^2} \simeq \int d^2z\,\frac{\bar{\varphi}_0(z,\bar{z})}{(1+|c|^2)^2} = 0 \tag{78}$$

as this is $\langle 1|\bar{\varphi}_0\rangle = 0$, i.e. $\bar{\varphi}_0$ is orthogonal to the constant zero mode of Δ. Thus we shall represent (77) by

$$\begin{aligned}\bar{m}^{-1}_{\bar{\mu}\beta}\left(\varphi_0,\psi_\beta\right) &= -\frac{\left(m'_{\bar{\alpha}\beta}\right)^T}{\bar{m}}\int d^2z\,\begin{pmatrix} c\Omega \\ cr\Gamma \\ r\Omega\end{pmatrix}_\beta \bar{\varphi}_0(z,\bar{z}) \\ &= -\frac{1}{\bar{m}}\int d^2z\,\begin{pmatrix} cA\Omega + rcB^\dagger\Gamma + rC^\dagger\Omega \\ cB\Omega + rcD\Gamma + rE^\dagger\Omega \\ cC\Omega + rcE\Gamma + rF\Omega\end{pmatrix}_\beta \bar{\varphi}_0(z,\bar{z})\end{aligned} \tag{79}$$

where Ω and Γ are functions of z, b and c. Similarly for the other inner product we need to calculate from (64)

$$(m^{-1}_{\bar{a}\bar{\alpha}}\bar{\varphi}_0,\bar{\psi}_{\bar{a}}) = \frac{m'_{\bar{\alpha}\beta}}{m}\int d^2w\,\frac{|w-b|^4}{(1+|w|^2)^2(|w-b|^2+|c|^2|w-b-r|^2)^2} \times \begin{pmatrix} -\frac{\bar{c}}{\bar{w}-\bar{b}} \\ \\ -\frac{\bar{c}\bar{r}}{(\bar{w}-\bar{b})^2} \\ \\ 1-\frac{\bar{r}}{\bar{w}-b}\end{pmatrix}_\beta \varphi_0(w,\bar{w})$$

we shall represent this as

$$\begin{aligned}(m^{-1}_{\bar{a}\bar{\alpha}}\bar{\varphi}_0,\bar{\psi}_{\bar{a}}) &= \frac{m'_{\bar{\alpha}\beta}}{m}\int d^2w\,\begin{pmatrix} \bar{c}\Omega' \\ \bar{c}\bar{r}\Gamma' \\ \bar{r}\Omega'\end{pmatrix}_\beta \varphi_0(w,\bar{w}) \\ &= -\frac{1}{m}\int d^2w\,\begin{pmatrix} \bar{c}A\Omega' + \bar{r}\bar{c}B\Gamma' + \bar{r}C\Omega' \\ \bar{c}B^\dagger\Omega' + \bar{r}\bar{c}D\Gamma' + \bar{r}E\Omega' \\ \bar{c}C^\dagger\Omega' + \bar{r}\bar{c}E^\dagger\Gamma' + \bar{r}F_2\Omega'\end{pmatrix}_\beta \varphi_0(w,\bar{w})\end{aligned} \tag{80}$$

where Ω' and Γ' are also functions of z, b and c. So using (79) and (80) then $\mathcal{J}$ is

$$\begin{aligned}\mathcal{J} &= \int d^2b \left[\oint_{|r|=\epsilon} d\bar{r}\, \frac{K^1}{|r|^2} \int d^2z \int d^2w\ \mathcal{Y}^r(z,w,r,b,c)\bar{\varphi}_0(z,\bar{z})\varphi_0(w,\bar{w}) \right. \\ &\quad \left. - \oint_{|r|=\epsilon} dr\, \frac{K^1}{|r|^2} \int d^2z \int d^2w\ \mathcal{Y}^{\bar{r}}(z,w,r,b,c)\bar{\varphi}_0(z,\bar{z})\varphi_0(w,\bar{w}) \right] \end{aligned} \tag{81}$$

where

$$\mathcal{Y}^r = \frac{1}{\bar{m}} \begin{pmatrix} cA\Omega + rcB^\dagger\Gamma + rC^\dagger\Omega \\ cB\Omega + rcD\Gamma + rE^\dagger\Omega \\ cC\Omega + rcE\Gamma + rF\Omega \end{pmatrix} \left(\frac{\partial}{\partial \bar{r}}, \frac{\partial}{\partial \bar{b}}, \frac{\partial}{\partial \bar{c}} \right) \frac{1}{m} \left(\bar{c}A\Omega' + \bar{r}\bar{c}B\Gamma' + \bar{r}C\Omega' \right) \tag{82}$$

$$\mathcal{Y}^{\bar{r}} = \frac{1}{\bar{m}} \left(cA\Omega + rcB^\dagger\Gamma + rC^\dagger\Omega \right) \left(\frac{\partial}{\partial r}, \frac{\partial}{\partial b}, \frac{\partial}{\partial c} \right) \frac{1}{m} \begin{pmatrix} \bar{c}A\Omega' + \bar{r}\bar{c}B\Gamma' + \bar{r}C\Omega' \\ \bar{c}B^\dagger\Omega' + \bar{r}\bar{c}D\Gamma' + \bar{r}E\Omega' \\ \bar{c}C^\dagger\Omega' + \bar{r}\bar{c}E^\dagger\Gamma' + \bar{r}F_2\Omega' \end{pmatrix} \tag{83}$$

To do the boundary integrals over r and $\bar{r}$ we set $r = \epsilon e^{i\theta}$ and $\bar{r} = \epsilon e^{-i\theta}$ so $dr = i\epsilon e^{i\theta} d\theta$ and $d\bar{r} = -i\epsilon e^{-i\theta} d\theta$. Only the θ-independent terms in the integrands contribute to $\oint d\theta \frac{1}{r}\mathcal{Y}^r$ and $\oint d\theta \frac{1}{\bar{r}}\mathcal{Y}^{\bar{r}}$. There is only one divergent piece from each term, and these pieces turn out to be the same. For $\oint d\theta \frac{1}{r}\mathcal{Y}^r$ we find

$$\frac{1}{r\bar{m}} \left(cA_1\Omega \right) \frac{\partial}{\partial \bar{r}} \left(\frac{1}{m} \left(\bar{c}A_1\Omega' \right) \right) = \frac{S^4\Omega\Omega'}{|c|^2\epsilon^2(ln(\epsilon^2))^3} \tag{84}$$

where A_1 and m are given in Appendix B. For $\oint d\theta \frac{1}{\bar{r}}\mathcal{Y}^{\bar{r}}$ we find an expression which is identical. So finally we obtain

$$\mathcal{J} = 2^6 e^{\gamma-2} \int d^2b\, d^2z\, d^2w\, \frac{(1+|b|^2)^4 \bar{\varphi}_0(z,\bar{z})\varphi_0(w,\bar{w})}{\epsilon^2(ln(\epsilon^2))^3|c|^2(z-b)(\bar{w}-\bar{b})} \tag{85}$$

which is our proposed counterterm. It is divergent as the zero instanton sector is approached from the one instanton sector as $\epsilon^2(ln(\epsilon^2))^3 \to 0$ as $\epsilon \to 0$. However the b integral is also infra-red divergent as a result of our working on the plane so as to be able to use the results of [5]. This problem will be addressed in the next section.

5 Conformal Invariance

The infra-red divergence of the topological counterterm for large values of the modulus $|b|$ can be approached by compactifying the spacetime onto a spherical surface by performing a Weyl transformation on the metric. This is analogous to the method used in Yang-Mills Theory [7] where a similar problem arises. We will show that the result of doing this is to render the integration over b heavily damped for large b.

The metric on the plane $ds^2 = d\bar{z}dz$ and on the sphere, $ds^2 = \Omega^2 d\bar{z}dz$ with $\Omega = 1 + \bar{z}z/h^2$, are related by a Weyl transformation, $g_{\mu\nu} \to e^{p(x)}g_{\mu\nu}$ with $p = ln\,\Omega^2$. This can be built out of infinitessimal transformations $\delta_p g_{\mu\nu} = \delta p g_{\mu\nu}$. The field w and quasi-ghost

ξ are independent of the metric, and the classical action $S[w]$ is Weyl invariant. Thus the Green's Function moduli density

$$g(t) = \int \mathcal{D}W \; e^{-\left(S[w]+(\varsigma+\tau^A\partial_A)(\xi^j F_j)\right)} \; \Lambda(w) \tag{86}$$

changes under the transformation δ_p such that

$$\delta_p g(t) = \int \mathcal{D}W \; e^{-S_{tot}} \left[\left(\int p\mathcal{M} - (\varsigma + \tau^A\partial_A)(\xi^j \; \delta_p F_j) \right) \; \Lambda(w) + \delta_p\Lambda(w) \right] \tag{87}$$

where the first term is from the action of δ_p on the volume element and $\mathcal{M}$ is the Weyl anomaly density. The last term can be removed provided $\delta_p\Lambda(w) = 0$ and as $(\varsigma + \tau^A\partial_A)\Lambda(w) = 0$ (ς doesn't act on w and w is independent of the t) then

$$\delta_p g(t) = \int \mathcal{D}W \; e^{-S_{tot}}\Lambda(w) \int p\mathcal{M} - \int \mathcal{D}W \; (\varsigma + \tau^A\partial_A)e^{-S_{tot}}(\xi^j \; \delta_p F_j)\Lambda(w) \tag{88}$$

However, the second term is linear in φ to first order in the expansion of $e^{-S_{tot}}$, thus it does not contribute until next to leading order. Also as $\int \mathcal{D}W \; e^{-S_{tot}} \; (\xi^j\delta_p F_j) \; \Lambda(w)$ is Grassmann odd,

$$\int \mathcal{D}W \; \varsigma \; e^{-S_{tot}} \; (\xi^j\delta F_j) \; \Lambda(w) = 0 \tag{89}$$

So terms in (87) from the constraint piece in the action do not contribute at the one loop level and we can write $\delta_p g(t)$ as

$$\delta_p \int \mathcal{D}W \; e^{-S[w]} = \int \mathcal{D}W \; e^{-S[w]} \int \delta p\mathcal{M} \tag{90}$$

Thus the behaviour of the Green's Function on the sphere is governed, to one loop, by the Weyl anomaly.

The calculation of the partition function involves the calculation of $\det\Delta$. However Δ has dimensions of mass so it is convenient to introduce an arbitrary parameter μ which also has dimensions of mass. To compensate for introducing μ the coupling constant becomes a function of μ, $k \to k(\mu)$. This means that the coupling is no longer scale invariant as it is classically. Now an infinitessimal global scaleing of the metric $g_{\mu\nu} \to g_{\mu\nu} + \lambda g_{\mu\nu}$, can be compensated by a shift in the mass-scale $\delta_\lambda\mu = -\frac{1}{2}\lambda\mu$ and $\delta_\lambda = \delta_\lambda\mu \; \frac{\partial}{\partial\mu} = -\frac{1}{2}\lambda\mu \; \frac{\partial}{\partial\mu}$. Thus the action $S(w)$ is no longer scale invariant either, since it contains the coupling. Using $S(w)$ from (28) so that now $S(w) = \frac{4}{k(\mu)} \int d^2x\hat{S}(w)$ then

$$\begin{aligned} \delta_\lambda S(w) &= -2\lambda \; \mu\frac{\partial}{\partial\mu}\left(\frac{1}{k(\mu)}\int d^2x\hat{S}(w)\right) \\ &= 2\lambda \; \mu \; \frac{1}{k(\mu)^2}\frac{\partial k(\mu)}{\partial\mu}\int d^2x\hat{S}(w) \end{aligned} \tag{91}$$

The renormalisation group β-function for the O(3) Sigma Model is given by $\beta = \mu\frac{\partial k}{\partial\mu}$, and is easily found to be $\beta = \frac{k^2}{4\pi}$. Thus

$$\int \lambda\mathcal{M} = -2\frac{\lambda}{k^2}\beta\int d^2x\hat{S}(w) \tag{92}$$

Hence

$$\mathcal{M} = -2\frac{\beta}{k^2}\hat{S}(w) = -\frac{1}{2\pi}\hat{S}(w) \tag{93}$$

up to total derivatives. Now, knowing $\mathcal{M}$, the Green's Function can be evaluated on the sphere using a position dependent scaleing p.

If $g_h(t)$ is the Green's Function moduli density and $\mathcal{D}_h W$ the functional integral volume element when the sigma model has as its space-time a sphere of radius h, then

$$\int dt\ g_h(t) = \int dt\ \mathcal{D}_h W\ e^{-S_{tot}}\Lambda(w) \tag{94}$$

if $\delta p = \delta h\ \frac{d}{dh}\ln\Omega^2$ then to one loop

$$\begin{aligned}\delta_p \int dt\ g_h(t) &= \int dt\ \mathcal{D}_h W\ e^{-S_{tot}}\left(\int \delta p\mathcal{M}\right)\Lambda(w)\\ &= \int dt\ \mathcal{D}_h W\ e^{-S_{tot}}\left(\int \delta h\ \left(\frac{d}{dh}\ln\Omega^2\right)\mathcal{M}\right)\Lambda(w)\end{aligned} \tag{95}$$

so

$$\frac{d}{dh}\int dt\ g_h(t) = \int dt\ \mathcal{D}_h W\ e^{-S_{tot}}\left(\int\left(\frac{d}{dh}\ln\Omega^2\right)\mathcal{M}\right)\Lambda(w) \tag{96}$$

Integrating with respect to h from h to infinity we obtain

$$\int dt\ g_h(t) = \int dt\ \mathcal{D}_\infty W\ \left[\exp\left(-S_{tot} + \int d^2x \ln\left(\frac{\Omega^2}{2}\right)\mathcal{M}\right)\right]\Lambda(w) \tag{97}$$

The extra term involving the Weyl anomaly suppresses the divergence due to the integral over b. Evaluating $\mathcal{M}$ at the classical solution gives $\hat{S}(w) = 8q'$ (see (30)) where q' is the topological charge density. For small $a-b$ the charge becomes concentrated at $z=b$ so that this density is approximately a delta-function $q' \propto \delta(z-b)$. Thus the additional contribution to the action due to the Weyl anomaly is

$$\begin{aligned}\int d^2x\ \ln\left(\Omega^2\right)q' &= \int d^2x\ \ln\left(\frac{2}{1+\frac{z^2}{h^2}}\right)^2\delta(z-b)\\ &= \int d^2x\ \ln\left(\frac{2}{1+\frac{b^2}{h^2}}\right)^2\end{aligned} \tag{98}$$

which, for large b, will supply a strong damping factor for the b-integration of Green's functions.

6 Conclusions

The moduli space integral of the $O(3)$ sigma model is divergent. To control the divergence we introduce a cut-off in moduli-space. The Green's functions then aquire a dependence on how the integration over field configurations is split into a quantum and background piece. It is essential that physical quantities should not possess such a dependence. To

study this we split the field in the one instanton sector by imposing a constraint on the quantum fluctuation and then varied Green's functions with respect to this constraint. The resulting 'anomaly' is an integral over the boundary of moduli-space resulting from the cut-off. It is in this region that one-instanton configurations degenerate to configurations in the zero-instanton sector. We found that the variation of the Green's function could be expressed as the variation of a zero-instanton sector expression. This provided us with a 'counter-term' with which to cancel the one-instanton moduli-space 'anomaly'.

This 'topological renormalisation' of the $O(3)$ sigma model is very similar to that of bosonic string theory [7]. There is certainly scope for further work on this phenomenon for other theories with pseudoparticle solutions.

Appendix A: Fluctuation operator and Greens Function

When we introduce the quantum fluctuation $\varphi(z,\bar{z})$ to the sigma model action of equation (30) we find that the second term can be expressed as

$$\frac{8}{k}\int d^2x \frac{\partial_{\bar{z}} w \partial_z \bar{w}}{(1+|w|^2)^2} = (\varphi, \Delta\varphi) \tag{99}$$

where

$$\Delta = -\frac{\rho^2}{\sqrt{g}}\partial_z \rho^{-2}\partial_{\bar{z}} \tag{100}$$

where $\partial_z = \frac{\partial}{\partial \bar{z}}$. This is different to the Δ found in [2]. The difference is because the fluctuation itself is defined differently in each case. It will be useful to put Δ in the form $\Delta = T^\dagger T$, where $T^\dagger$ is the adjoint of T with respect to our inner product (37) so $T = \frac{1}{g^{\frac{1}{4}}}\partial_{\bar{z}}$ and $T^\dagger = -\frac{\rho^2}{\sqrt{g}}\partial_z g^{\frac{1}{4}}\rho^{-2}$. The Green's Function of Δ is the two-point Green's Function for φ, and satisfies

$$\Delta\mathcal{I}(x,y) = \frac{\rho^2}{\sqrt{g}}\,\delta^2(x-y) - P(x,y) \tag{101}$$

where $P(x,y)$ is a projection operator To find the form of $P(x,y)$ we look at the properties of $\mathcal{I}$. If $\mathcal{Z}(x)$ is a zero mode of T (and again we use a dot notation to indicate an inner product), then $\bar{\mathcal{Z}}(x).\Delta\mathcal{I}(x,y) = (T\mathcal{Z}, T\mathcal{I}(x,y)) = 0$ so $\bar{\mathcal{Z}}(x).P(x,y) = \bar{\mathcal{Z}}(y)$. Also, φ is constrained by our choice of F to be orthogonal to ψ. So the two point function of φ and $\varphi.\psi$ must vanish, hence $\mathcal{I}(x,y).\psi(y) = 0$ which leads us to $P(x,y).\psi(y) = \psi(x)$, thus we can deduce that

$$P(x,y) = \psi_a(x)\bar{m}^{-1}_{\bar{b}a}\bar{\mathcal{Z}}_{\bar{b}}(y) \tag{102}$$

We shall show that the one instanton sector Green Function is

$$\mathcal{I}(x,y) = -\frac{1}{\pi^2}\int d^2z\,(1-P^\dagger.)\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}}(1-.P) \tag{103}$$

with $P(x,y)$ as above, and $P^\dagger(x,y) = \mathcal{Z}_b(x) m^{-1}_{b\bar{a}} \bar{\psi}_{\bar{a}}(y)$. The dot notation, as above indicates an inner product. So writing this out explicitly gives

$$
\begin{aligned}
\mathcal{I}(x,y) &= -\frac{1}{\pi^2}\int d^2z \left[\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}}\right. \\
&\quad - \int d^2x'\; P^\dagger(x,x')\sqrt{g}\;\rho^{-2}(x')\frac{1}{x'-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y}} \\
&\quad - \int d^2y'\;\frac{1}{x-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y'}}\sqrt{g}\;\rho^{-2}(y')P(y',y) \\
&\quad \left. - \int d^2x'\; d^2y'\; P^\dagger(x,x')\sqrt{g}\;\rho^{-2}(x')\frac{1}{x'-z}\rho^2(z)\frac{1}{\bar{z}-\bar{y'}}\sqrt{g}\;\rho^{-2}(y')P(y',y)\right]
\end{aligned}
$$

The deriavatives in Δ act only on the x variable, and as $\frac{\partial}{\partial\bar{x}}\frac{1}{x-z} = \pi\;\delta^2(x-z)$ then

$$
\begin{aligned}
\frac{\partial}{\partial\bar{x}}\mathcal{I}(x,y) &= -\frac{1}{\pi}\int d^2z \left[\delta^2(x-z)\rho^2(z)\frac{1}{\bar{z}-\bar{y}}\right. \\
&\quad \left. - \int d^2y'\;\delta^2(x-z)\rho^2(z)\frac{1}{\bar{z}-\bar{y'}}\sqrt{g}\;\rho^{-2}(y')P(y',y)\right] \\
&= -\frac{1}{\pi}\int d^2x\frac{\rho^2(x)}{\bar{x}-\bar{y}} + \frac{1}{\pi}\int d^2x\; d^2y'\frac{\rho^2(x)}{\rho^2(y')}\frac{\sqrt{g}}{\bar{x}-\bar{y'}}P(y',y) \qquad (104)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial x}\;\rho^2(x)\;\frac{\partial}{\partial\bar{x}}\;\mathcal{I}(x,y) &= -\int d^2x\;\delta^2(\bar{x}-\bar{y}) \\
&\quad + \int d^2x\; d^2y'\;\delta^2(\bar{x}-\bar{y'})\frac{\sqrt{g}}{\rho^2(y')}P(y',y) \\
&= -1 + \int d^2x\;\frac{\sqrt{g}}{\rho^2(x)}P(x,y) \qquad (105)
\end{aligned}
$$

which gives us (101).

In the zero instanton sector the Green Function becomes

$$\mathcal{I}_0(x,y) = -\frac{1}{\pi^2}\int d^2z\;(1-\Pi^\dagger.)\frac{1}{x-z}\rho_0^2(z)\frac{1}{\bar{z}-\bar{y}}(1-.\Pi) \qquad (106)$$

where Π is the zero mode projector in the zero instanton sector

$$\Pi f = \frac{\int\sqrt{g}\;f}{\int\sqrt{g}}\quad,\quad \rho_0 = 1+|c|^2 \qquad (107)$$

Π and P both project constant functions onto themselves since these are zero-modes in both sectors. Hence $\Pi.P = \Pi$. We need to find the relationship between $\mathcal{I}_0(x,y)$ and $\mathcal{I}(x,y)$. First notice that

$$(1-P^\dagger.)\;\mathcal{I}_0(x,y)\;(1-.P) = -\frac{1}{\pi^2}\int d^2z\;(1-P^\dagger.)\frac{1}{x-z}\rho_0^2(z)\frac{1}{\bar{z}-\bar{y}}(1-.P) \qquad (108)$$

Then for ϵ small

$$\rho = \rho_0 + O(\epsilon) \qquad (109)$$

so

$$\mathcal{I}(x,y) = (1-P^\dagger.)\;\mathcal{I}_0(x,y)\;(1-.P)\times(1+O(\epsilon)) \qquad (110)$$

Appendix B: Kähler Metric

Here we give a brief review of the calculation to find the one instanton Kähler Metric used above. The metric tensor on the space of instanton parameters, found in [5], is

$$m_{AB} = \int d^2x \sqrt{g} \left(\frac{\partial \bar{v}}{\partial t^A} \frac{\partial v}{\partial t^B} \right) \frac{1}{(1+|v|^2)^2} \tag{111}$$

In a particular topological sector this can be written in terms of a Kähler potential $\mathcal{K}$ since

$$\begin{aligned} \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^{\bar{\beta}}} \int d^2x \ \sqrt{g} \ \ln(1+|v|^2) &= \frac{\partial}{\partial t^{\bar{\beta}}} \int d^2x \ \sqrt{g} \ \left(\frac{1}{(1+|v|^2)} \bar{v} \frac{\partial v}{\partial t^\alpha} \right) \\ &= \int d^2x \sqrt{g} \left(\frac{\partial \bar{v}}{\partial t^{\bar{\beta}}} \frac{\partial v}{\partial t^\alpha} \right) \frac{1}{(1+|v|^2)^2} \end{aligned} \tag{112}$$

which has the same form as the metric tensor given above. The metric $g_{\mu\nu}$ for the sphere S^2_{phys} is

$$g_{\mu\nu} = \delta_{\mu\nu} \left(1+|z|^2\right)^{-2} \tag{113}$$

so

$$\sqrt{g} = \left(1+|z|^2\right)^{-2} \tag{114}$$

thus

$$m_{\alpha\bar{\beta}} = \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^{\bar{\beta}}} \mathcal{K} \tag{115}$$

where

$$\mathcal{K} = \int d^2x \ \left(1+|z|^2\right)^{-2} \ \ln(1+|v|^2) \tag{116}$$

which can be calculated. As $z = x + iy$ and the one instanton solution is

$$v = c \left(\frac{z-a}{z-b} \right) \tag{117}$$

as before, then

$$\mathcal{K} = \int dx \ dy \ (1+|z|^2)^{-2} \left[ln(|z-b|^2 + |c|^2 |z-a|^2) - ln(|z-b|^2) \right] \tag{118}$$

However, the second logarithm introduces a divergence in the limit $z \to b$. This can be seen by looking at the double differentiation of the second logarithm with respect to b and $\bar{b}$.

$$\frac{\partial}{\partial b} \frac{\partial}{\partial \bar{b}} ln(|z-b|^2) = \frac{1}{\pi} \delta^2(z-b) \tag{119}$$

This problem can be solved by noticing that as

$$\frac{1}{(1+|v|^2)^2} \frac{\partial v}{\partial b} \frac{\partial \bar{v}}{\partial \bar{b}} = \frac{|c|^2 |z-a|^2}{(|z-b|^2 + |c|^2 |z-a|^2)^2} = \frac{\partial}{\partial b} \frac{\partial}{\partial \bar{b}} \left(ln(|z-b|^2 + |c|^2 |z-a|^2) \right) \tag{120}$$

then we can ignore the second logarithm. We are left with

$$\mathcal{K} = \int dx \ dy \ (1+|z|^2)^{-2} \left[ln(|z-b|^2 + |c|^2 |z-a|^2) \right] \tag{121}$$

This integral can be solved by using two tricks to put the integrand into an exponential form

$$\int_0^\infty d\alpha\ \alpha\ \exp\left(-\alpha(1+|z|^2)\right) = \left(1+|z|^2\right)^{-2} \tag{122}$$

and

$$ln(m) - ln(n) = \int_0^\infty \frac{dt}{t}\ \left(e^{-mt} - e^{-nt}\right) \tag{123}$$

giving

$$\begin{aligned}\mathcal{K} &= \int d\alpha\ dt\ dx\ dy \frac{\alpha}{t}\Big[\exp\left(-(x^2+y^2)(\alpha + At) + 2xtB + 2ytC - Dt - \alpha\right) \\ &\qquad\qquad - \exp\left(-\alpha(x^2+y^2) - \alpha - t\right)\Big]\end{aligned} \tag{124}$$

where we have put

$$A = (1+|c|^2),\ B = b_1 + a_1|c|^2,\ C = b_2 + a_2|c|^2,\ D = |b|^2 + |a|^2|c|^2 \tag{125}$$

so a and b have been split into their real and imaginary parts ($a = a_1 + ia_2$ etc.). The integrals over x and y can be done as simple Gaussians, leaving

$$\mathcal{K} = \int d\alpha\ dt\ \frac{\alpha}{t}\left[\frac{\pi}{\alpha + At}\exp\left(\frac{(B^2+C^2)t^2}{\alpha + At} - (Dt + \alpha)\right) - \frac{\pi}{\alpha}\exp(-\alpha - t)\right] \tag{126}$$

To progress we change variables to λ and t where $\alpha = \lambda t$, $d\alpha = t\ d\lambda$ and

$$\begin{aligned}\mathcal{K} &= \int d\lambda\ dt\ \pi\left[\frac{\lambda}{\lambda + A}\exp\left(t\left[\frac{(B^2+C^2)}{\lambda + A} - (D+\lambda)\right]\right) - \exp\left(-t\left(\lambda + 1\right)\right)\right] \\ &= \int d\lambda\ \pi\left[\frac{\lambda}{\lambda^2 + (A+D)\lambda + AD - (B^2+C^2)} - \frac{1}{\lambda+1}\right]\end{aligned} \tag{127}$$

The first term is just a standard integral of the form

$$\begin{aligned}\int_0^\infty \frac{\tau}{\zeta + \eta\tau + \theta\tau^2}d\tau &= \frac{1}{2\theta}ln(\zeta + \eta\tau + \theta\tau^2) \\ &\quad - \frac{\eta}{2\theta(\eta^2 - 4\zeta\theta)^{\frac{1}{2}}}ln\left(\frac{\eta + 2\theta\tau - (\eta^2 - 4\zeta\theta)^{\frac{1}{2}}}{\eta + 2\theta\tau + (\eta^2 - 4\zeta\theta)^{\frac{1}{2}}}\right)\end{aligned} \tag{128}$$

The solution evaluated in the limit $\lambda \to \infty$ is zero. So with

$$G = \frac{4(AD - B^2 - C^2)}{(A+D)^2} \tag{129}$$

$$\mathcal{K} = \frac{\pi}{2}\left[\frac{1}{(1-G)^{\frac{1}{2}}}\ ln\left(\frac{1-(1-G)^{\frac{1}{2}}}{1+(1-G)^{\frac{1}{2}}}\right) - ln(AD - B^2 - C^2)\right] \tag{130}$$

To simplify this we can make the change of variables $a = b + r$, where we are interested in the limit $r \to 0$. In this case $AD - B^2 - C^2 = |c|^2|a-b|^2 = |c|^2|r|^2$, so g is small and K] can be expanded in powers of $|r|^2$. Terms of order $|\epsilon|^4$ can be dropped and

$$\begin{aligned}\mathcal{K} &= \pi\left[-ln\left(A+D\right) + \frac{G}{4}lnG - \left(1 + \frac{G}{2}\right)ln2 + \frac{G}{8}\right] \\ &= \pi\left[-lnS - \frac{|c|^2}{S^2}(b\bar{r} + \bar{b}r)\frac{|c|^2|r|^2}{S^2}\left(\frac{1}{2}(1-2S) + 2ln\left(\frac{|c|^2}{S}\right) + ln|r|^2\right)\right]\end{aligned} \tag{131}$$

where $S=(1+|b|^2)(1+|c|^2)$. This can now be differentiated to find $m_{\alpha\bar{\beta}}$. Remember that we are now working with coordinates $\{t^\alpha\}=\{r,b,c\}$ and $\{t^{\bar{\beta}}\}=\{\bar{r},\bar{b},\bar{c}\}$.

$$m_{\alpha,\bar{\beta}}=\pi\begin{pmatrix}\mathcal{A} & \mathcal{B} & \mathcal{C}\\ \mathcal{B}^\dagger & \mathcal{D} & \mathcal{E}\\ \mathcal{C}^\dagger & \mathcal{E}^\dagger & \mathcal{F}\end{pmatrix} \tag{132}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{|c|^2}{S^2}\left[\frac{5}{2}-S+\mathcal{V}\right]\\
\mathcal{B} &= -\mathcal{H}\left[S+\bar{r}b\left(5-S+2\mathcal{V}\right)\right]\\
\mathcal{C} &= -\frac{c\bar{b}(1+|b|^2)}{S^2}+\frac{c\bar{r}(1+|b|^2)}{S^3}\left[\frac{7}{2}-\frac{3|c|^2}{2}-S+(1-|c|^2)\mathcal{V}\right]\\
\mathcal{D} &= -\frac{1}{(1+|b|^2)^2}+2(b\bar{r}+\bar{b}r)\mathcal{H}\\
\mathcal{E} &= \frac{c}{S^2}(\bar{b}^2r-\bar{r})\\
\mathcal{F} &= -\frac{1}{(1+|c|^2)^2}\left[1+\frac{(b\bar{r}+\bar{b}r)}{S}(1-|c|^2)\right]\\
\mathcal{H} &= \frac{|c|^2(1+|c|^2)^2}{S^3}\quad,\quad \mathcal{V}=ln\left(\frac{|c|^4|\epsilon|^2}{S^2}\right)
\end{aligned} \tag{133}$$

The inverse matrix given in (72) is

$$m^{-1}_{\bar{\alpha}\beta}=\frac{1}{m}\begin{pmatrix}A & B & C\\ B^\dagger & D & E\\ C^\dagger & E^\dagger & F\end{pmatrix} \tag{134}$$

In the limit $r\to 0$ the determinant is

$$m=\det m=\frac{|c|^2}{S^4}\,ln|r|^2 \tag{135}$$

and the ϵ dependence of the components is

$$\begin{aligned}
A &= A_1+rA_2+\bar{r}A_3+|r|^2A_4+r^2A_5+\bar{r}^2A_6\\
B &= B_1+rB_2+\bar{r}B_3+|r|^2B_4+r^2B_5+r\ln|r|^2B_6+|r|^2\ln|r|^2B_7+r^2\ln|r|^2B_8\\
C &= C_1+rC_2+\bar{r}C_3+|r|^2C_4+r^2C_5+r\ln|r|^2C_6+|r|^2\ln|r|^2C_7+r^2\ln|r|^2C_8\\
D &= \ln|r|^2D_0+D_1+rD_2+\bar{r}D_3+|r|^2D_4+\bar{r}\ln|r|^2D_5\\
&\qquad +r\ln|r|^2D_6+|r|^2\ln|r|^2D_7+|r|^2(\ln|r|^2)^2D_8\\
E &= E_1+rE_2+\bar{r}E_3+|r|^2E_4+\bar{r}\ln|r|^2E_5+r\ln|r|^2E_6+|r|^2\ln|r|^2E_7+|r|^2(\ln|r|^2)^2E_8\\
F &= \ln|r|^2F_0+F_1+rF_2+\bar{r}F_3+|r|^2F_4+\bar{r}\ln|r|^2F_5\\
&\qquad r\ln|r|^2F_6+|r|^2\ln|r|^2F_7+|r|^2(\ln|r|^2)^2F_8
\end{aligned} \tag{136}$$

where the A_i's, B_i's, C_i's, D_i's, E_i's and F_i's are all just functions of b and c. The term used in (84) is

$$A = \frac{1}{\mathcal{S}^2} \tag{137}$$

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